# On the behavior of positive solutions of semilinear elliptic equations in asymptotically cylindrical domains

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**Abstract**

The goal of this note is to study the asymptotic behavior of positive solutions for a class of semilinear elliptic equations which can be realized as minimizers of their energy functionals. This class includes the Fisher-KPP and Allen-Cahn nonlinearities. We consider the asymptotic behavior in domains becoming infinite in some directions. We are in particular able to establish an exponential rate of convergence for this kind of problems.

## 1 Introduction

Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\mathcal{D}$ and consider the semilinear elliptic problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathcal{D}, \\ u > 0 & \text{in } \mathcal{D}, \\ u = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{1.1}$$

It is a classical fact that Problem 1.1 has a solution $0 < u < 1$ provided that $f$ is of class $C^1([0,1])$ and satisfies the following assumptions:

$$f(0) = 0 = f(1), \quad f(s) > 0 \quad \text{for all} \quad s \in (0,1). \tag{1.2}$$

$$f'(0) > \lambda_1(\mathcal{D}) \tag{1.3}$$

where $\lambda_1(\mathcal{D})$ is the first eigenvalue of $-\Delta$ under Dirichlet boundary conditions, given by

$$\lambda_1(\mathcal{D}) = \inf_{u \in H_0^1(\mathcal{D})} \frac{\int_{\mathcal{D}} |\nabla u|^2}{\int_{\mathcal{D}} u^2}.$$

This can be seen using barriers: $\bar{u} \equiv 1$ is a supersolution and $\underline{u} = \varepsilon\phi_1$ is a subsolution of (1.1) with $\underline{u} \leq \bar{u}$ provided that $\varepsilon > 0$ is sufficiently small, and $\phi_1$ is a positive eigenfunction of $-\Delta$ associated to $\lambda_1(\mathcal{D})$. See for instance Hess [13], Clement-Sweers [10], de Figueiredo [11]. In addition, the solution $0 < u < 1$ is unique provided that $f$ satisfies the additional assumption

$$f'(s) < \frac{f(s)}{s} \quad \text{for all} \quad s \in (0,1), \tag{1.4}$$

as established by Brezis-Oswald in [2]. All these assumptions are automatically satisfied for the Fisher-KPP or Allen-Cahn nonlinearities

$$f(u) = \lambda u(1-u), \quad f(u) = \lambda u(1-u^2),$$

if $\lambda > \lambda_1(\Omega)$.

In what follows we assume that $f \in C^1([0,1])$ satisfies assumptions (1.2), (1.3) and (1.4).

Let $\omega \subset \mathbb{R}^k$ be a bounded, smooth convex domain with $0 \in \omega$. For a positive number $\ell$ we let

$$\Omega_\ell := \ell\omega \times \mathcal{D} \subset \mathbb{R}^{n+k} \tag{1.5}$$

and consider the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_\ell, \\ u > 0 & \text{in } \Omega_\ell, \\ u = 0 & \text{on } \partial\Omega_\ell. \end{cases} \tag{1.6}$$

We observe that

$$\lambda_1(\Omega_l) = \lambda_1(\mathcal{D}) + \ell^{-2}\lambda_1(\omega)$$

and hence assumption (1.3) will be satisfied in $\Omega_\ell$ for $\ell$ sufficiently large. We deduce the existence of a unique solution $0 < u_\ell < 1$ to (1.6) for all large $\ell$.

The purpose of this paper is to analyze the behavior as $\ell \to +\infty$ of the solution $u_\ell$, in connection with the unique solution $0 < u_{\mathcal{D}} < 1$ of (1.1). Our main result is the following.

**Theorem 1.1.** *For all* $(X_1, X_2) \in \mathbb{R}^k \times \bar{\mathcal{D}}$ *we have*

$$u_\ell(X_1, X_2) \to u_{\mathcal{D}}(X_2) \quad \text{as} \quad \ell \to +\infty,$$

*uniformly in compact subsets of* $\mathbb{R}^k \times \overline{\mathcal{D}}$. *Moreover this local convergence is exponential: there exists a positive number* $\alpha$ *such that*

$$u_{\mathcal{D}}(X_2) - e^{-\alpha\ell} \leq u_\ell(X_1, X_2) \leq u_{\mathcal{D}}(X_2)$$

*for all* $(X_1, X_2) \in \frac{\ell}{2}\omega \times \bar{\mathcal{D}}$.

The solutions $u_\ell$ and $u_{\mathcal{D}}$ can be variationally characterized as follows. First we observe that with no loss of generality we may assume that $f(s) = 0$ for all $s \geq 1$ or $s \leq 0$ since a solution under this assumption automatically satisfies $0 \leq u \leq 1$ thanks to the maximum principle. We let

$$F(s) = -\int_0^s f(t)dt.$$

Then $u$ solves (1.1) if and only if $u$ is the unique nontrivial critical point of the functional

$$E_{\mathcal{D}}(u) = \frac{1}{2}\int_{\mathcal{D}} |\nabla u|^2 + \int_{\mathcal{D}} F(u), \quad u \in H_0^1(\mathcal{D}).$$

This functional has a global minimizer since it is coercive and lower semi-continuous. This global minimizer is nontrivial since $E(\varepsilon\phi_1) < 0$ for all small $\varepsilon > 0$ thanks to assumption (1.3), and hence it characterizes the solution $u_{\mathcal{D}}$. A similar characterization of course holds true for $u_\ell$.

The question of analyzing the behavior of minimizers of various variational problems passing from truncated to infinite cylindrical domains, in terms of minimizers for their cross sections has been treated in in [7, 6, 5, 3, 4, 8]. In the current context we take strong advantage of the Euler equation to establish comparisons. Some of the arguments we use are present in the analysis of solutions with helicoidal symmetries of the Allen-Cahn equation in [12, 9].

We devote the rest of this paper to the proof of Theorem 1.1.

## 2 Asymptotic behaviour

First we prove the following comparison principle, which is adapted from the uniqueness result of Brezis-Oswald [2], see also [1]. For this, assume $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary.

**Lemma 2.1.** *Let $0 < u_1, u_2 < 1$ be functions in $H^1(\Omega)$ such that in a weak sense*

$$\begin{cases} \Delta u_1 + f(u_1) \geq 0 = \Delta u_2 + f(u_2) \ \textit{in } \Omega, \\ \qquad u_1 \leq u_2 \ \textit{on } \partial\Omega. \end{cases} \tag{2.1}$$

*Then one has $u_1 \leq u_2$ in $\Omega$.*

*Proof.* Let $\theta \in C^\infty(\mathbb{R})$ be such that

$$\theta'(t) \geq 0, \quad \theta(t) = 0 \text{ for } t \leq 0, \quad \theta(t) = 1 \text{ for } t \geq 1.$$

Set $\theta_\varepsilon(t) = \theta(\frac{t}{\varepsilon})$. One has

$$\theta_\varepsilon(u_1 - u_2) \in H_0^1(\Omega).$$

Multiplying the left hand side of the first line of (2.1) by $u_2$, the right hand side by $u_1$, subtracting we get

$$-u_2\Delta u_1 - u_2 f(u_1) + u_1\Delta u_2 + u_1 f(u_2) \leq 0.$$

Multiplying then by $\theta_\varepsilon(u_1-u_2)$ and integrating over $\Omega$ we get

$$\begin{aligned}
&\int_\Omega (u_1 f(u_2)-u_2 f(u_1))\theta_\varepsilon(u_1-u_2)dx \le \int_\Omega (u_2\Delta u_1 - u_1\Delta u_2)\theta_\varepsilon(u_1-u_2)dx\\
&\quad = -\int_\Omega u_2|\nabla(u_1-u_2)|^2\theta'_\varepsilon(u_1-u_2)dx + \int_\Omega \nabla u_2\cdot\nabla(u_1-u_2)\theta'_\varepsilon(u_1-u_2)(u_1-u_2)dx\\
&\quad \le \int_\Omega \nabla u_2\cdot\nabla(u_1-u_2)\theta'_\varepsilon(u_1-u_2)(u_1-u_2)dx.
\end{aligned}$$

Let us set

$$\gamma_\varepsilon(t)=\int_0^t s\theta'_\varepsilon(s)ds.$$

Then the inequality above reads if $\{u_1>u_2\}=\{x\in\Omega \mid u_1(x)>u_2(x)\}$

$$\begin{aligned}
\int_{\{u_1>u_2\}} u_1u_2(\frac{f(u_2)}{u_2}-\frac{f(u_1)}{u_1})\theta_\varepsilon(u_1-u_2)dx &\le \int_\Omega \nabla u_2\cdot\nabla\gamma_\varepsilon(u_1-u_2)dx\\
&= \int_\Omega -\Delta u_2\,\gamma_\varepsilon(u_1-u_2)dx.
\end{aligned}$$

It is clear that

$$0\le\gamma_\varepsilon(t)\le\int_0^\varepsilon s\theta'(\frac{s}{\varepsilon})\frac{1}{\varepsilon}ds\le C\varepsilon.$$

Since $\Delta u_2$ is bounded passing to the limit above leads to

$$\int_{\{u_1>u_2\}} u_1u_2(\frac{f(u_2)}{u_2}-\frac{f(u_1)}{u_1})dx\le 0.$$

Since $\frac{f(u)}{u}$ is decreasing thanks to assumption (1.4), it follows that $\{u_1>u_2\}$ as measure zero. This completes the proof. $\square$

The points in $\mathbb{R}^k\times\mathbb{R}^n$ are denoted by

$$x=(X_1,X_2),\quad X_1\in\mathbb{R}^k,\quad X_2\in\mathbb{R}^n.$$

When necessary, we will denote by $\Delta_{X_2}$ the Laplacian in $x_2$ and similarly by $\nabla_{X_1}$, $\nabla_{X_2}$ the gradients in $X_1$, $X_2$.

In what follows, $\Omega_\ell$ is the domain (1.5) and $u_\ell$ is the solution of (1.6). The hypothesis that $\omega$ is a convex domain containing the origin implies that if $0<\ell\le\ell'$ then $\ell\omega\subset\ell'\omega$.

**Lemma 2.2.** *Suppose that $\ell$ is large enough so that $f'(0)>\lambda_1(\Omega_\ell)$. Then for any $\ell'>\ell$ one has*

$$0<u_\ell\le u_{\ell'}<1 \quad in\ \ \Omega_\ell. \tag{2.2}$$

*Moreover when $\ell\to\infty$*

$$u_\ell\to u_{\mathcal{D}}$$

*in $C^{1,\alpha}_{loc}(\mathbb{R}^k\times\overline{\mathcal{D}})$.*

*Proof.* On $\Omega_\ell$ the functions $u_\ell,\ u_{\ell'}$ are both positive solutions to

$$\Delta u + f(u) = 0. \tag{2.3}$$

We assume here that the functions are extended by 0 outside of $\Omega_\ell$ or $\Omega_{\ell'}$. The inequality (2.2) follows from Lemma 2.1. Since the sequence of functions $u_\ell$ is monotone and bounded above, the pointwise limit

$$u_\infty(X_1, X_2) = \lim_{\ell\to\infty} u_\ell(X_1, X_2).$$

exists. Moreover, from $u_\ell \leq 1$, for any $\ell_0 > 0$ the $H^1(\Omega_{\ell_0})$-norm of $u_\ell$ is bounded independently of $\ell$. Therefore $u_\infty \in H^1_{loc}(\mathbb{R}^k \times \mathcal{D})$ and it vanishes on $\mathbb{R}^k \times \partial\mathcal{D}$.

We would like to show now that $u_\infty$ is independent of $X_1$. For $i = 1, \cdots, k$ we set

$$\tau^i_h v(x) = v(x - he_i), \quad h > 0,$$

where $e_i$ denotes the $i$-th vector of the canonical basis of $\mathbb{R}^k \times \mathbb{R}^n$. We claim that

$$u_{\ell+h} \geq \tau^i_{h'} u_\ell \quad \text{for } 0 < h' \leq \lambda h \tag{2.4}$$

$\lambda \leq 1$ being so small that

$$\lambda e_i \in \omega. \tag{2.5}$$

Indeed if (2.5) holds, we have for $X_1 - h'e_i \in \ell\omega$ and some $Y_1 \in \omega$

$$X_1 = \ell y_1 + h'e_i = (\ell + h)\{\frac{\ell}{\ell + h} Y_1 + \frac{h}{\ell + h}\frac{h'}{h} e_i\} \in (\ell + h)\omega$$

(since $y_1, \frac{h'}{h} e_i \in \omega$ and $\omega$ is a convex set containing 0). Thus the support of $\tau^i_{h'} u_\ell$ is contained in $\Omega_{\ell+h}$.

Then, on this support, $\tau^i_{h'} u_\ell$ and $u_{\ell+h}$ are both solution to (2.3). Since $u_{\ell+h}$ is positive $u_{\ell+h} \geq \tau^i_{h'} u_\ell$ on the boundary of this support and (2.4) follows from Lemma 2.1. Similarly, one would get

$$\tau^i_{-h'}(u_\ell) \leq u_{\ell+h}.$$

Thus, passing to the limit in $\ell$ in the inequalities above one derives

$$u_\infty(x - h'e_i) \leq u_\infty(x), \quad u_\infty(x + h'e_i) \leq u_\infty(x),$$

which implies

$$u_\infty(x) \leq u_\infty(x - h'e_i) \leq u_\infty(x),\ \forall i = 1, \cdots, k,\ \forall h' \text{ small.}$$

This shows that $u_\infty$ is independent of $X_1$.

Since $u_\ell$ vanishes on $\ell_0\omega_1 \times \partial\mathcal{D}$ so does $u_\infty$ and therefore $u_\infty \in H^1_0(\mathcal{D})$. Passing to the limit in the equation

$$-\Delta u_\ell + f(u_\ell) = 0 \quad \text{in } \Omega_{\ell_0}$$

one gets

$$-\Delta u_\infty + f(u_\infty) = 0 = -\Delta_{X_2} u_\infty + f(u_\infty) \quad \text{in } \Omega_{\ell_0},$$

where, as we mentioned above, $\Delta_{X_2}$ denotes the Laplace operator in $\mathbb{R}^n$. It follows that $u_\infty = u_{\mathcal{D}}$ by uniqueness of the solution $0 < u < 1$ of (1.1).

The convergence in $C^{1,\alpha}_{loc}$ follows from the Schauder estimates. □

We have shown that $u_\ell \to u_{\mathcal{D}}$ when $\ell \to \infty$ in $C^{1,\alpha}_{loc}(\mathbb{R}^k \times \overline{\mathcal{D}})$. However, for this kind of problems one expects an exponential rate of convergence. This is what we would like to establish now.

If $0 < u_{\mathcal{D}} < 1$ is the unique solution of (1.1) we denote by $\mu_1$ the first eigenvalue of the Dirichlet problem

$$-\Delta\phi - f'(u_{\mathcal{D}})\phi = \mu\phi, \quad \phi \in H^1_0(\mathcal{D}) \tag{2.6}$$

and by $\varphi_1$ its corresponding positive eigenfunction normalized so that its $L^2(\mathcal{D})$-norm is equal to 1.

Let us first show.

**Lemma 2.3.** *One has*

$$\mu_1 > 0. \tag{2.7}$$

*Proof.* Multiplying (1.1) by $\varphi_1$ and integrating in $\mathcal{D}$, we get,

$$0 = \int_{\mathcal{D}} (f'(u_{\mathcal{D}})u_{\mathcal{D}}\varphi_1 + \mu_1 u_{\mathcal{D}}\varphi_1 - f(u_{\mathcal{D}})\varphi_1)dX_2.$$

Thus

$$\mu_1 \int_{\mathcal{D}} u_{\mathcal{D}}\varphi_1 dX_2 = \int_{DD} (f(u_{\mathcal{D}}) - f'(u_{\mathcal{D}})u_{\mathcal{D}}\ )\varphi_1 dX_2 > 0,$$

by (1.4). Since $u_{\mathcal{D}}$ and $\varphi_1$ are both positive on $\mathcal{D}$, (2.7) follows. $\square$

## Proof of Theorem 1.1

Since $\omega$ contains the origin there exists an hypercube $Q_c = (-c, c)^k$ such that

$$Q_c \subset \omega,$$

and thus

$$\ell Q_c \subset \ell\omega.$$

Denote by $0 < \tilde{u}_\ell < 1$ the solution of (1.6) in $\tilde{\Omega}_\ell = \ell Q_c \times \omega_2$. One has obviously by our previous comparison theorem

$$u_\ell \geq \tilde{u}_\ell. \tag{2.8}$$

We consider then $\varphi_1 = \varphi_1(X_2)$ the positive eigenfunction of (2.6) normalized so that $\|\varphi_1\|_{L^2(\mathcal{D})} = 1$, and

$$w_\kappa(X_1) = \sum_{i=1}^{k} \frac{\cosh(\sigma x_i)}{\cosh(\sigma(\ell - \kappa))},$$

where $\sigma$ and $\kappa$ are positive constants that we will choose later on. Set

$$\underline{u}(X_1, X_2) = u_{\mathcal{D}}(X_2) - \varepsilon\varphi_1(X_2)w_\kappa(X_1) = u_\infty - \varepsilon\varphi_1 w_\kappa.$$

One has on $\tilde{\Omega}_{\ell-\kappa}$

$$\Delta\underline{u} + f(\underline{u}) = \Delta u_{\mathcal{D}} - \varepsilon w_\kappa \Delta\varphi_1 - \varepsilon\varphi_1\Delta w_\kappa + f(u_{\mathcal{D}} - \varepsilon\varphi_1 w_\kappa).$$

Since

$$f(u_{\mathcal{D}} - \varepsilon\varphi_1 w_\kappa) = f(u_{\mathcal{D}}) - f'(u_{\mathcal{D}})\varepsilon\varphi_1 w_\kappa - \int_{u_{\mathcal{D}}-\varepsilon\varphi_1 w_\kappa}^{u_{\mathcal{D}}} (f'(t) - f'(u_\infty))dt,$$

we obtain,

$$\Delta\underline{u} + f(\underline{u}) = \varepsilon w_\kappa \varphi_1(\mu_1 - \sigma^2) + I_\varepsilon \tag{2.9}$$

where

$$I_\varepsilon = -\int_{u_\infty-\varepsilon\varphi_1 w_\kappa}^{u_\infty} (f'(t) - f'(u_\infty))dt.$$

It is clear that $0 \leq w_\kappa \leq k$ on $\tilde{\Omega}_{\ell-\kappa}$. Thus due to the uniform continuity of $f'$, one has for some $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$

$$|I_\varepsilon| \leq \varepsilon\delta(\varepsilon)\varphi_1 w_\kappa.$$

Going back to (2.9) we deduce

$$\Delta\underline{u} + f(\underline{u}) \geq 0 \quad \text{in } \tilde{\Omega}_{\ell-\kappa}$$

for

$$\sigma^2 < \mu_1 \text{ and } \varepsilon \text{ small enough,} \tag{2.10}$$

that is, $\underline{u}$ is a subsolution to the equation $\Delta u + f(u) = 0$ in $\tilde{\Omega}_{\ell-\kappa}$. We will suppose from now on that $\sigma$ and $\varepsilon$ are fixed and satisfy (2.10). Note that on any compact subset of $\mathbb{R}^k$, $w_\kappa$ converges exponentially towards 0. If one can show that

$$\tilde{u}_\ell \geq \underline{u} \quad \text{ on } \partial\tilde{\Omega}_{\ell-\kappa} \tag{2.11}$$

by Lemma 2.1 one will have $\tilde{u}_\ell \geq \underline{u}$ on $\tilde{\Omega}_{\ell-\kappa}$ and thus by (2.8) the theorem will follow.

To prove (2.11) it is enough to show that

$$\tilde{u}_\ell \geq \underline{u} = u_{\mathcal{D}} - \varepsilon\varphi_1 w_\kappa \text{ on } \partial(\ell-\kappa)Q_c \times \mathcal{D},$$

since on the rest of the boundary of $\tilde{\Omega}_{\ell-\kappa}$ both functions are vanishing. Since on $\partial(\ell-\kappa)Q_c \times \mathcal{D}$ one has $w_\kappa \geq 1$, it is enough to show that

$$\tilde{u}_\ell \geq u_{\mathcal{D}} - \varepsilon\varphi_1 \text{ on } \partial(\ell-\kappa)Q_c \times \mathcal{D}.$$

Suppose that we have shown that

$$\tilde{u}_\kappa(0, X_2) \geq u_{\mathcal{D}}(X_2) - \varepsilon\varphi_1(X_2) \quad \text{ on } \mathcal{D}, \tag{2.12}$$

for some $\kappa < \ell$. Let $\overline{x}$ denote a point on $\partial(\ell-\kappa)Q_c$. One has for some $i = 1, \ldots, k$

$$\overline{X} = (\overline{x}_1, \ldots, \ell-\kappa, \ldots, \overline{x}_k)$$

where $\ell - \kappa$ occupies the $i^{th}$-slot, $|\overline{x}_j| \leq \ell - \kappa$ for any other $j \neq i$. Since the equations at stakes are invariant by translation one has clearly

$$\tilde{u}_\ell(x) \geq \tilde{u}_\kappa(X_1 - \overline{X}, X_2)$$

on the support of this last function which is clearly contained in $\tilde{\Omega}_\ell$ and thus the above inequality holds in $\tilde{\Omega}_\ell$ (see Lemma 2.1). For $x = (\overline{X}, X_2)$ which is on $\partial(\ell - \kappa)Q_c \times \mathcal{D}$ one has then

$$\tilde{u}_\ell(\overline{X}, X_2) \geq \tilde{u}_\kappa(0, X_2) \geq u_{\mathcal{D}}(X_2) - \varepsilon\varphi_1(X_2),$$

that is, $\tilde{u}_\ell \geq u_\infty - \varepsilon\varphi_1$ on $\partial(\ell - \kappa)Q_c \times \mathcal{D}$. Thus we are reduced to prove (2.12) for some $\kappa < \ell$.

Let us denote by $\nu$ the inner unit normal to $\partial\mathcal{D}$ and by $D_\delta$ the set

$$D_\delta = \{x \in \mathcal{D} \mid x = x_0 + \lambda\nu, x_0 \in \partial\mathcal{D}, \lambda \in (0, \delta)\}$$

for some $\delta > 0$ small so that $D_\delta$ is contained in $\mathcal{D}$. Due to the Hopf maximum principle, the positivity and continuity of $\varphi_1$, there exists a positive number $m$ such that for $\delta$ small one has

$$\frac{\varphi_1(x_0 + \lambda\nu)}{\lambda} \geq m \quad \forall x = x_0 + \lambda\nu \in D_\delta.$$

Since for some positive constant $A$ one has $\varphi_1 \geq A$ on $\mathcal{D}\backslash D_\delta$, one has for $\kappa$ large

$$\tilde{u}_\kappa(0, X_2) \geq u_{\mathcal{D}} - \varepsilon A \geq u_{\mathcal{D}}(X_2) - \varepsilon\varphi_1(X_2) \quad \text{ on } \mathcal{D}\backslash D_\delta, \tag{2.13}$$

because $\tilde{u}_\kappa(0, \cdot) \to u_{\mathcal{D}}$ uniformly in $\mathcal{D}$ as $\kappa \to \infty$.

On the other hand for $x_0 + \lambda\nu \in D_\delta$ one has

$$\frac{\tilde{u}_\kappa(0, x_0 + \lambda\nu)}{\varphi_1(x_0 + \lambda\nu)} = \frac{u_{\mathcal{D}}(x_0 + \lambda\nu)}{\varphi_1(x_0 + \lambda\nu)} + \frac{\tilde{u}_\kappa(0, x_0 + \lambda\nu) - u_{\mathcal{D}}(x_0 + \lambda\nu)}{\varphi_1(x_0 + \lambda\nu)}$$

and

$$\begin{aligned} \frac{|\tilde{u}_\kappa(0, x_0 + \lambda\nu) - u_{\mathcal{D}}(x_0 + \lambda\nu)|}{\varphi_1(x_0 + \lambda\nu)} &= \frac{|\int_0^\lambda \frac{d}{dt}(\tilde{u}_\kappa(0, x_0 + t\nu) - u_{\mathcal{D}}(x_0 + t\nu))dt|}{\lambda} \frac{\lambda}{\varphi_1(x_0 + \lambda\nu)} \\ &\leq \operatorname*{Max}_{t \in (0,\delta)} |\nabla_{x_2}\tilde{u}_\kappa(0, x_0 + t\nu) - \nabla_{x_2}u_{\mathcal{D}}(x_0 + t\nu)| \frac{1}{m} \\ &\leq \varepsilon \end{aligned}$$

by the $C^{1,\alpha}$ convergence of $\tilde{u}_\kappa(0, x_2)$ toward $u_{\mathcal{D}}(x_2)$, for $\kappa$ large enough. From this inequality one derives

$$\frac{\tilde{u}_\kappa(0, x_0 + \lambda\nu)}{\varphi_1(x_0 + \lambda\nu)} \geq \frac{u_{\mathcal{D}}(x_0 + \lambda\nu)}{\varphi_1(x_0 + \lambda\nu)} - \varepsilon \quad \forall \ (x_0 + \lambda\nu) \in D_\delta$$

which reads also

$$\tilde{u}_\kappa(0, x_0 + \lambda\nu) \geq u_{\mathcal{D}}(x_0 + \lambda\nu) - \varepsilon\varphi_1(x_0 + \lambda\nu) \quad \forall \ (x_0 + \lambda\nu) \in D_\delta.$$

Combining this and (2.13) we arrive to (2.12) which completes the proof of the theorem.

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